Chapter 9

Symmetry in quantum mechanics

9.1 Introduction

Right and so we begin the 'symmetry in quantum mechanics' part of the course. My aim here is to give you a little taste of group and representation theory and its relevance to quantum mechanics. You will not learn much new actual physics in this section - but I hope to try and convey the deep mathematics underlying many quantum phenomena that you have already seen (e.g., the presence of degeneracy, the addition of angular momentum, and, if we have time, dephasing). Group and representation theory is a massive area of mathematics and I will only scratch the surface - my main aim is to make it *relatively* friendly to leave you comfortable with the basic ideas and keen to learn more for future courses/projects! One thing that is perhaps worth highlighting here is group theory appears in all sorts of places you don't expect it. For example, it can be used to encode symmetries for more efficient machine learning models. Therefore, even if you are fed up of physics and cannot wait to get a job earning money as a software developer - this part of the course could be very useful for you.

These notes are driven by the pedagogical philosophy that most people learn best by examples and intuition. Therefore, throughout these notes I try as hard as I can to provide examples and more informal handwavey explanations of the key ideas wherever possible. In places I have sacrificed some formality to do so. I have also for the sake of time relegated some of the longer proofs to the appendices. If you are interested in a more formal presentation of this material (and more that I will not cover) I have uploaded to moodle Vincenzo Savona's old notes (in French and English). However, I hope my notes will prove helpful to those of you who also like examples and an attempt at more wordy explanations. For those of you that do like a 'physicist-approach-to-maths' I cannot recommend enough Group Theory In A Nutshell For Physicists. It's longer and more detailed that you will need for this course but a very friendly read. I'll try and point out useful sections where relevant. Lie Algebras for Physics is also good but its even more detailed than needed.

9.1.1 Motivational examples: Symmetry is everywhere in quantum!

Spatial translations. By way of introduction let's start with a simple example considering spatial translations that I have borrowed from Terry Rudolph. Suppose I asked you to write down a wavefunction $\psi(x)$ that is invariant under arbitrary translations in x, i.e. $x \to x + a$ for any a. What could you write down?

Intuitively if it's anything other than constant in x then the function will not be spatially invariant, i.e. we've got to have $\psi(x)$ = constant. In Terry's words - *It is questionable whether this is valid - is it normalizable for example? But imagine we plough ahead like good physicists and ignore the mathematical difficulties.* If we Fourier transform this wavefunction then we get that this wavefunction can be written in the momentum basis as $\phi(p) = \delta(p)$ (the Fourier transform of the constant function is a delta function).

But is this the only function that is invariant under spatial translations? What if we instead consider a function of the form $\psi(x) = e^{ipx}$? Then we see that translating $x \to x + a$ produces only an extra "overall phase" of e^{ipa}. This is a global phase and so doesn't change anything physical about the state. That is, the state is (up to a non-physical global phase) also invariant under translations. If we again Fourier transform to the momentum representation we now have $\psi(x) = e^{ip'x}$ is $\phi(p) = \delta(p - p')$, so this is a state of fixed definite momentum p'. That is, momentum is conserved in this translationally invariant state.

In Terry's words again we learn two things from this example: *(i) that we should only expect a small subset of the possible quantum states to obey a particular symmetry, and (ii) that there can be an intimate connection between a particular observable (momentum) and that symmetry.*

Now imagine we have prepared one of these translationally invariant states, e.g. a momentum eigenstate. Under what Hamiltonian evolutions will it remain translationally invariant/a momentum eigenstate? Intuitively we need any potential $V(x)$ to also be translationally invariant, otherwise this will break the initial translational symmetry. This means the only potential Hamiltonian is the free particle Hamiltonian $H = \frac{1}{2m}\hat{p}^2$. Or, more concretely, we require that

$$[e^{-i\hat{p}b}, \hat{H}] = 0. \tag{2}$$

which will be true for any Hamiltonian such that $[\hat{p}, \hat{H}] = 0$. Thus we see that *the property of translational symmetry is associated with 'conservation of momentum'.*

A similar story could be told about the relationship between rotational invariance and angular momentum. And both of these cases are symptomatic of a much deeper story about the intimate connection between conservation laws and stuff that commutes with a Hamiltonian and symmetries. This can be made precise and of sweeping generality in Noether's theorem. But let's start with the basics and pin down a more general mathematical formalism to discuss symmetries.

The mystery of degeneracy. When you diagonalize a generic Hermitian matrix, the eigenvalues will in general be distinct. But for physical Hamiltonians of quantum systems we often find that the eigenvalues are degenerate - that is there are distinct eigenstates with the same eigenenergy. In the early days of quantum mechanics this was somewhat surprising! We will see that this can be explained by symmetry properties of a system.

Consider a unitary transformation U that leaves H invariant, i.e.,

$$U^\dagger H U = H\,. \tag{9.1}$$

Or, equivalently, we have

$$HU = UH\,. \tag{9.2}$$

Given that U commutes with H we have that if $H|\psi\rangle = E|\psi\rangle$, then

$$HU|\psi\rangle = UH|\psi\rangle = UE|\psi\rangle = EU|\psi\rangle\,. \tag{9.3}$$

That is, the action of U on a state $|\psi\rangle$ produces an eigenstate of H with the same energy E. Or, in other words, U produces another degenerate eigenstate.

More generally, given a family of transformations $\{U\}$ there will be a corresponding family of degenerate energy eigenstates. That is, the presence of a symmetry gives rise to degeneracy. We will see later how group and representation theory will provide a means of predicting/explaining the number of degenerate states.

9.1.2 Introduction to groups

A symmetry describes some property of a system, i.e. some function f or of some dataset $\mathbb{R}$, which is left unchanged under some transformation. As we are, for the purpose of this course, predominantly interested in quantum systems, let's suppose that the transformation refers to a *unitary evolution*[1] applied to the quantum state, i.e., to a map $\rho \to U\rho U^\dagger$ for some U. Now crucially, such symmetry transformations form *a group*.

Proposition 9.1.1. *Let G be the set of all unitary symmetry transformations, such that for any $U \in G$, the map $\rho \to U\rho U^\dagger$ leaves some property of ρ unchanged. Then, G, equipped with multiplication, forms a group.*

What is a *group*?

Definition 9.1.2. A group is a set equipped with an operation that combines any two elements to form a third element while being associative as well as having an identity element and inverse elements.

Formally, one can write a set G equipped with the operation "$*$" is a group if one has:

- G is closed under the operation $*$. That is, if $a \in G$ and $b \in G$ then $a * b \in G$.
- Associativity: $\forall a, b, c \in G$, one has $(a * b) * c = a * (b * c)$.
- An identity element: There exists an element $e \in G$ such that $e * a = a \ \forall a \in G$. Such an element is unique and is called the identity of the group.
- Inverse element: $\forall a \in G$, it exists $b \in G$ such that $b * a = a * b = e$. We then say that $b = a^{-1}$. For each e the element a^{-1} is unique and is called the inverse of a.

How can we see that any unitary that leaves a property invariant forms a group with $*$ matrix multiplication (i.e. that Proposition 9.1.1 is true)? With a little thought we can see that each of the defining properties of a group are satisfied.

- Closure: Given any two unitaries U and V in G, the unitary $V * U$ obtained by multiplying V and U is also a symmetry transformation. This follows from the fact that concatenating two property-preserving transformations $\rho \to U\rho U^\dagger \to V * U\rho U^\dagger * V^\dagger$ constitutes in itself a property-preserving transformation.
- Associativity: for any unitaries U, V, W we have $U(VW) = (UV)W$.
- Identity element: Clearly the identity matrix I leaves any property of a state unchanged and for any unitary we have $IU = U$ and so I is indeed the identity element e.

[1]We will encounter and work with symmetry representations that are ostensibly not unitary. However, a wide class of representations are equivalent to unitary ones. In particular, Wigner's theorem guarantees that all symmetry transformations of quantum states preserving inner products are either unitary or antiunitary, and often antiunitary transformations are "unitary and complex conjugation".

- Inverse: For each U in G, there exists an element $U^\dagger$ in G such that $U * U^\dagger = U^\dagger * U = I$, where I is the identity matrix, and $U^\dagger$ is the inverse (conjugate transpose) of U because if U conserves some property, then U^{-1} also conserves that property.

In broad terms *groups* encode abstract symmetries, and *representations* describe concrete realisations of those symmetries in physical systems. In most maths courses people learn about groups first before moving onto representations later. However, in practise, in everyday physics we often identify symmetries at the level of the representation and then "abstractify" them: i.e. connect a familiar physical symmetry with some familiar abstract mathematical group.

To quote Representation Theory for Geometric Quantum Machine Learning: *"The main utility of this abstractification procedure is that groups as mathematical objects have been thoroughly studied since the early 19th century, and a wealth of information is readily available for scores of them. Moreover, in the eyes of physics, the list of abstract groups is surprisingly short, thanks in large part to classification programs for finite groups and semisimple Lie groups—and nature's seeming preferential treatment of these groups—this means that identification is direct in many cases."* That is, if you have a physics (or perhaps even a classical machine learning) problem and can identify the relevant group - odds are some long dead mathematician has already half solved your problem and so you can save yourself a lot of work.

In broad terms a representation is a map from the elements of a group to a set of unitaries[2] such that the unitaries obey the same properties under composition as the original group. We will define this more formally later but I just wanted to mention it informally now because I think it helps to understand why we care about groups in the first place- the key point being often in practise we will identify the representation first and then abstractify to find the underlying group and then plug in centuries of maths to help us understand it better.

9.1.3 Finite group examples

Groups can be either finite or continuous. Let's consider some examples of finite groups first.

Definition 9.1.3 (Finite group)**.** A group that contains a finite number of element is called a finite group. The number of element is called the *order* of the group.

One way to uniquely identify a group is via its Cayley table. Named after the 19th century British mathematician Arthur Cayley, a Cayley table describes the structure of a finite group by arranging all the possible products of all the group's elements in a square table reminiscent of an addition or multiplication table. Many properties of a group can be discovered from its Cayley table.

Order 1 group. The only group with only one element is the trivial group containing just the identity element, e.g. $G = e$. Its Cayley table can be written as:

$$\begin{array}{c|c} * & e \\ \hline e & e \end{array} \tag{9.4}$$

A possible representation of this group is $e \to I$.

Order 2 group. The unique Cayley table for a group with only two elements is the group where the only non-identity element is its own inverse element, e.g. $G = e, a$ such that $aa^{-1} =$

[2]Representations need not strictly be unitary but essentially all the ones we'll care about here will be.

$aa = e$, i.e.

$*$	e	a
e	e	a
a	a	e

(9.5)

One possible group with this Cayley table is $G = \{1, -1\}$ with $*$ standard scalar multiplication. (In this case, the map $e \to 1$ and $a \to -1$ is a representation of the group)

Other examples include the groups composed of $G = \{I, X\}$, $G = \{I, Z\}$ and $G = \{I, \text{SWAP}\}$ with $*$ matrix multiplication. (In this case, the maps $e \to I$, $a \to X$ and $e \to I$, $a \to Z$ and $e \to I$, $a \to \text{SWAP}$ are representations of the group).

Another possible group with the same Cayley table is the parity group that contains the "transformation in the mirror" that turns x into $-x$. Let us define the operator $\hat{P}$ such that $\hat{P}f(x) = \hat{P}f(-x)$. Given $\hat{P}\hat{P} = 1$, we see that the set of transformation $\{1, \hat{P}\}$ form a group.

All of these groups are isomorphic (share the same Cayley table) to the $\mathbb{Z}_2$ group (cyclic group on 2 elements). The Cayley table captures the fundamental symmetry but it can manifest in different ways.

Order 3 group. The unique (it might not be obvious now that it is unique - we will come back to this in a bit) Cayley table for a group with only three elements is the $\mathbb{Z}_3$ group (cyclic group with three elements):

$*$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

(9.6)

An example of such group is the set of 2D rotations that leave a triangle invariant. Or the 3rd roots of unity in the complex plane $a_j = e^{i2\pi \frac{j}{3}}$ equipped with multiplication.

Order 4 groups. Again we can consider the cyclic group $\mathbb{Z}_4$

$*$	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

(9.7)

An example of such group is the set of 2D rotations that leave a square invariant. Or the 4th roots of unity in the complex plane.

But 4th order is also the smallest order that is not unique. That is, there is another possible Cayley table for a group of four elements that is not isomorphic (i.e. the same up to relabeling) as the Cayley table above:

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

(9.8)

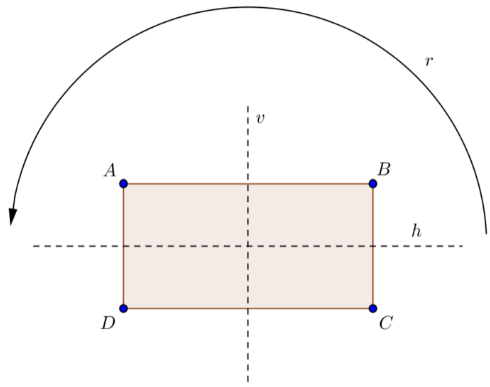

Figure 9.1: Diagram of the symmetry group of a rectangle (dihedral group R_2): (Wiki page on the Dihedral group).)

Note that here each element is its own inverse but there are cyclic transformations between a, b and c. An example of such a group would be the symmetries of a rectangle as sketched in Fig. 9.1. The group elements are identity e, rotation r (in either direction) by π and reflections h and v about the horizontal and vertical axes respectively.

Order 6 groups. Again we can consider the cyclic group $\mathbb{Z}_6$. Alternatively we can have:

$*$	e	a	a^2	b	c	d
e	e	a	a^2	b	c	d
a	a	a^2	e	c	d	b
a^2	a^2	e	a	d	b	c
b	b	d	c	e	a^2	a
c	c	b	d	a	e	a^2
d	d	c	b	a^2	a	e

(9.9)

This is called the C3v group.. This is the symmetry group of a triangle as shown in Fig. 9.2. There are 6 possible transformations that leave the triangle invariant:

- The identity e which leaves all coordinates unchanged.
- The proper rotation c_+ by an angle of $2\pi/3$ in the positive trigonometric sense (i. e. counter-clockwise). And the clockwise version c_-.
- Reflection along each axis (there are three of them).

See Fig. 9.2 for a sketch of this. The C3v also captures the symmetry of the Ammonia molecule, NH3. There will be a question on the problem sheet this week on this. This will be one of our favorite example groups so its worth becoming very familiar with it.

Other important (larger) finite groups include:

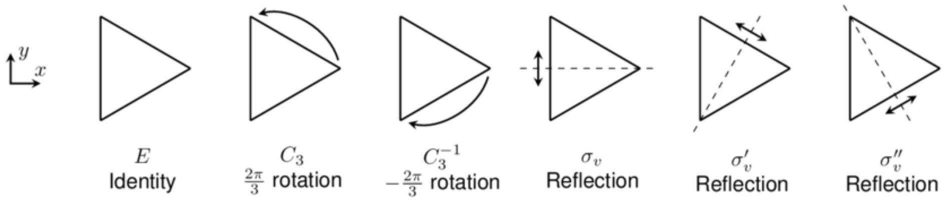

Figure 9.2: Diagram of the symmetry group of a triangle (C3v). Note that I used the notation $c_+ = C_3$ and $c_- = c_-$ to denote the rotations but this image uses C_3 and c_-. I took this image from (Fundamental properties of 2D excitons bound to single stacking faults in GaAs).)

The cyclic group $\mathbb{Z}_n$. For completeness, of course we can also consider the cyclic group of n objects $\mathbb{Z}_n$

$$\begin{array}{c|ccccc} * & e & a_1 & a_2 & \dots & a_{n-1} \\ \hline e & e & a_1 & a_2 & \dots & a_{n-1} \\ a_1 & a_1 & a_2 & a_3 & \dots & e \\ a_2 & a_2 & a_3 & a_4 & \dots & a_1 \\ \vdots & & & & & \\ a_{n-1} & a_{n-1} & e & a_1 & \dots & a_{n-2} \end{array} \tag{9.10}$$

Examples of such groups include the set of 2D rotations that leave a regular n-sided polygon invariant and the nth roots of unity $a_j = e^{i2\pi\frac{j}{n}}$ in the complex plane.

Symmetric permutation group S_n. The group is composed of the group of all possible permutations of n object with the group operation the composition of functions.

As there are $n!$ such permutations operations the order of the symmetric group is $n!$

For example, $S_3 = \{I, \mathrm{SWAP}_{12}, \mathrm{SWAP}_{13}, \mathrm{SWAP}_{23}, \mathrm{CYCLE}_{123}, \mathrm{CYCLE}_{321}\}$. (What is the CAYLEY table for this group? [3])

This is a very important group in quantum physics as (as we saw earlier) it is the symmetry group of systems of indistinguishable particles.

9.1.4 Continuous group examples

A non-finite group is a continuous group. Of particular importance are *Lie* groups [4].

Definition 9.1.4 (Lie group)**.** Informally, a Lie group is a continuous group that depends *analytically* on some continuous parameters λ.

We list some important examples of Lie groups below.

[3] Hint we have already seen that there are only two possible tables for an order 6 group

[4] Note that not all infinite groups are Lie groups! The set of all rational numbers equipped with addition is infinite (but countable), but it is not a Lie group. But again, we're physicists not mathematicians and all the continuous groups we'll care about (at least in this course) will be Lie groups.

Real d-dimensional rotations $SO(d)$. A classic example of a Lie group is the group of all rotation matrices (i.e. orthogonal matrices with determinant 1) for real d dimensional rotation vectors. An orthogonal matrix is the real analogue of a unitary matrix and is defined by the properties $\Re[M] = M$ and $MM^T = M^T M = I$. For an orthogonal matrix to be a rotation matrix we also require that $det(M) = 1$.

For example, the elements of the group $SO(2)$ (i.e. rotation matrices in $2D$) can be written as

$$M(\phi) = \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix}. \tag{9.11}$$

Another commonly encountered case is $SO(3)$ which corresponds to all rotations in $3D$.

The orthogonal group $O(d)$. Another example of a continuous group is $O(d)$ which is simply the group of orthogonal matrices (i.e. without the restriction that the determinant of the matrices equals 1). Orthogonal matrices preserve the inner product between real vectors $\langle x'|y'\rangle = (\langle x|O^T)(O|y\rangle) = \langle x|O^T O|y\rangle = \langle x|y\rangle$. They thus correspond to rotations and reflections.

Note that the determinant of any orthogonal matrix is +1 or −1. This follows from $1 = \det(I) = \det(M^T M) = \det(M^T)\det(M) = (\det(M))^2$. Orthogonal matrices with a -1 determinant can implement reflections, e.g.

$$M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{9.12}$$

performs a reflection of the vector (x, y) in the y-plane.

The unitary group $U(d)$. $U(d)$ is the group of $d \times d$ dimensional unitary matrices. This is the group of matrices that preserve the length/inner product of quantum states.

For example, $U(1)$ can be represented just as the unit circle in the complex plane $[e^{i\phi}]$. Or it can be represented as a rotation around any single axis on the Bloch sphere, e.g. $[R_z(\phi)]$ where $R_z(\phi) = e^{-i\phi Z}$.

Similarly, $U(2)$ represents all 2-dimensional unitaries, that is all unitaries on a single qubit. We recall that any single qubit unitary can be written as

$$R(\boldsymbol{n}, \theta, \phi) = e^{-i(\phi I + \theta \boldsymbol{n}.\boldsymbol{\sigma})} \tag{9.13}$$

where we stress that for full generality we need to include the global phase term generated by ϕI. However, this global phase is unphysical. This motivates the consideration of instead the special unitary group.

The special unitary group $SU(d)$. $SU(d)$ corresponds to the group of unitary matrices with determinant 1. The restriction to determinant 1 effectively fixes the arbitrary global phase. To see this note that multiplying a unitary matrix by a phase matrix $e^{-i\phi}I$ manifests as a change in the phase of its determinant as $\det(e^{-i\phi}IM) = \det(e^{-i\phi}I)\det(M) = e^{-id\phi}\det(M)$.

For example $SU(2)$ corresponds to the group of unitary rotations to a single qubit that can be written as

$$R(\boldsymbol{n}, \theta) = e^{-i\theta \boldsymbol{n}.\boldsymbol{\sigma}}\,. \tag{9.14}$$

Recall that this can be represented as the set of rotations of the Bloch vector of a state on the Bloch sphere. This would seem to be *in some sense* equivalent to the group $SO(3)$, i.e. the group of real rotations in 3D. Indeed the groups $SU(2)$ and $SO(3)$ are very closely related - more on this in a bit.

9.2 Basic definitions and properties of groups

Now that you're equipped with a whole zoo of examples let's go back to looking at the basic mathematical structure of groups and some of their most important properties.

Definition 9.2.1 (Abelian and non-Abelian groups)**.** : If $a * b = b * a \ \forall a, b \in G$, the group G is said to be Abelian. Otherwise it is called a non-Abelian group. These groups are also called commutative and non-commutative.

For example, $U(1)$ is Abelian (phases commute) but $U(2)$ is not (arbitrary unitaries do not commute). As we will see later, whether or not a group is Abelian effects some of their most fundamental properties. (In particular, Abelian groups tend to be much simpler to study).

Another important concept is that of a subgroup.

Definition 9.2.2 (Subgroup)**.** A subset H of the group G is a subgroup of G if and only if it is nonempty and itself forms a group.

The closure conditions mean the following: Whenever a and b are in H, then $a * b$ and a^{-1} are also in H. These two conditions can be combined (*exercise: show this!*) into one equivalent condition: whenever a and b are in H, then $a * b^{-1}$ is also in H. The identity of a subgroup is the identity of the group: if G is a group with identity e_G, and H is a subgroup of G with identity e_H, then $e_H = e_G$.

Definition 9.2.3 (Proper Subgroup)**.** We call a subgroup of G which is neither the identity nor G itself a *proper* subgroup.

A fundamental result in the theory of finite groups is Lagrange theorem:

Theorem 9.2.4 (Lagrange)**.** *Let G be a finite group and H a subgroup of G, then the order of H (i.e. the number of its elements) divides the order of G.*

We prove this theorem in sec.9.10.2.

It is easy to see that this implies in particular that if the order of a group is prime then there is only one possible group (i.e. one unique Cayley table) for that group. To see this note that if the order n of a finite group G is a prime, then it has no divisors, and so no subgroups. The only group with no proper subgroups is the cyclic one Z_n for prime n - so this is the unique group. Recall that I claimed earlier that $\mathbb{Z}_3$ was the unique group with 3 elements - this is why.

Let's look back at the non-cycle 4th order group we discussed earlier with the Cayley table:

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

(9.15)

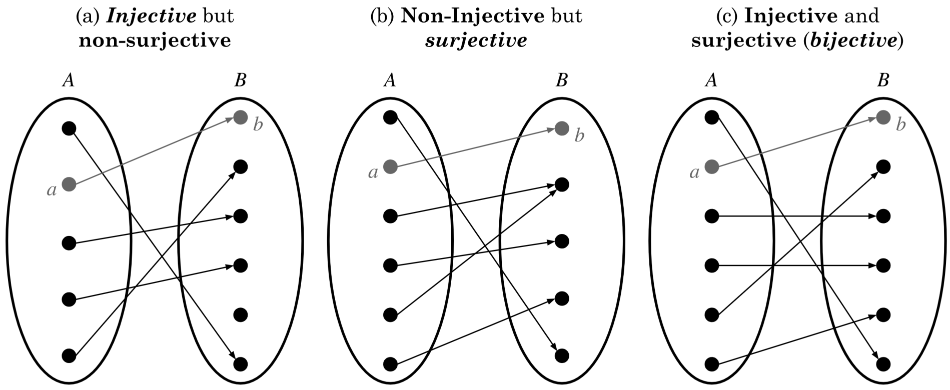

Figure 9.3: Diagram of injective, surjective and bijective functions: (Wiki page on functions).)

This has subgroups $\{e, a\}$ and $\{e, b\}$ and $\{e, c\}$ which are all $\mathbb{Z}_2$ groups. Or, thinking more physically and recalling that this corresponds to the symmetries of a rectangle as sketched in Fig. 9.1, identity and any one of the transformations (e.g. rotation by π, reflection in the horizontal axis, reflection in the vertical axis) each forms a group because each of these transformations are self-inverse.

Exercise: What are the subgroups of C3v group? Does this make sense in terms of the symmetries of the Ammonia molecule, NH3?

Group Homomorphism and isomorphism The final important concept I will discuss in this section is that of group homomorphisms and isomorphisms. This formalises the important idea that I have been repeatedly hinting at but glossing over - the idea of superficially different looking groups being the same in some sense.

A group homomorphism, is a mapping between two groups which respects the group structure:

Definition 9.2.5 (Group homomorphism)**.** A function from a group $(G, *)$ to the group $(G', \star)$ is an application $f : G \to G'$ such that $\forall x, y \in G \quad f(x * y) = f(x) \star f(y)$. (This implies that $f(e) = e'$, where e and e' denote the respective neutrals of G and G' and $f(x^{-1}) = f(x)^{-1}$.)

For instance, it is always possible to create a morphism of any finite group to the trivial group by mapping all the elements to e'. A less trivial example is that the group Z_2 is homomorphic to $Z = \{\ldots, -3, -2, -1, 0, 1, 2, \ldots\}$ equipped with addition using $f(x) = 1$ for even numbers and $f(x) = -1$ for odd numbers for $x \in Z$.

A homomorphism from $f : G \to G'$ can be bijective, i.e. be a map with a one-to-one correspondence between elements in the domain and range as sketched in Fig. 9.11. In this case, we call the mapping an isomorphism.

Definition 9.2.6 (Group isomorphism)**.** A group isomorphism is a function between two groups that sets up a one-to-one correspondence between the elements of the groups in a way that respects the given group operations.

If there exists an isomorphism between two groups, then the groups are called isomorphic. From the standpoint of group theory, isomorphic groups have the same properties and need not be distinguished. In the case of finite groups, this means that the groups have the same Cayley table.

For example, $G = \{1, -1\}$ with $*$ standard scalar multiplication, $G = \{I, X\}$ or $G = \{I, \mathrm{SWAP}\}$ with $*$ matrix multiplication are isomorphic to $\mathbb{Z}_2$. Similarly, multiplication on the unit circle in the complex plane $[e^{i\phi}]$ and rotation around any single axis on the Bloch sphere, e.g. $[R_z(\phi)]$ where $R_z(\phi) = e^{-i\phi Z}$, are isomorphic to $U(1)$. (However, Z_2 is homomorphic, but not isomorphic, to Z equipped with addition).

9.3 Basic definitions and properties of representations

Let us now return to representations. As I mentioned earlier *groups* encode abstract symmetries but *representations* describe concrete realisations of those symmetries. Informally, a representation of a group captures the action of a group on a vector space (e.g. on quantum states). In particular, in a quantum context, it is a map from the elements of a group to a set of unitaries such that multiplication of that set of unitaries obeys the same properties as the original group. For example, the group Z_2 can be represented as $\{1, X\}$ and $\{1, \text{SWAP}\}$ acting on C^2 and $(C^2)^{\otimes 2}$ respectively. We can formally define the notion of a representation of a group via the notion of homomorphisms introduced above.

Definition 9.3.1 (Group representation)**.** A representation R of a group G on a vector space V is a group homomorphism [5] from G to a set of matrices that act on a vector space V. The dimension of a representation R is defined to be the dimension of the vector state V, i.e., $\dim(R) = \dim(V)$.

We can think of this pictorially as:

$$\begin{array}{ccccccc} g_1 & \cdot & g_2 & = & g_1 & \cdot & g_2 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ D(g_1) & \cdot & D(g_2) & = & D(g_1 & \cdot & g_2) \end{array}$$

where $D(g)$ is a $d \times d$ dimensional matrix that acts on a d dimensional vector space V.

We stress that formally a representation is by definition the *map R*. However, more informally the word representation is used in multiple ways. For example, informally you might hear someone discuss the $\{1, \text{SWAP}\}$ representation of $\mathbb{Z}_2$. Technically $\{1, \text{SWAP}\}$ is a group (that is isomorphic to $\mathbb{Z}_2$) and the representation is the map R such that $R(e) = I$ and $R(a) = \text{SWAP}$ (where the properties of a and e are captured by the $\mathbb{Z}_2$ Cayley table). As long as you remember that fundamentally it is the underlying map that is the representation, this casual way of speaking shouldn't cause too much confusion in practise[6].

Let us give a few examples:

Trivial representation. All groups admit a trivial representation (or the Identity representation): $\forall g \in G, R(g) = I$.

Examples representations for the parity group $Z_2 = \{e, a\}$.

- As we said before we have the representations $G = \{1, X\}$ and $G_{\text{SWAP}} = \{1, \text{SWAP}\}$ acting on C^2 and $(C^2)^{\otimes 2}$ respectively. You could also have[7] $G = \{1, Z\}$ on C^2.
- On $\mathbb{R}$ it has two representations: 1) the trivial representation $R(g) = 1$ for $g = e, a$, as well as 2) the representation $R(e) = 1, R(a) = -1$.

[5] in most cases we will look at it will also be an isomorphism, i.e., a one-to-one map

[6] This subtlety is put nicely in Representation Theory for Geometric Quantum Machine Learning: *As an unfortunate feature of the subject, the word "representation" can equivalently refer to the group homomorphism R, the vector space upon which it acts V, or the image subgroup $R(G) \subset GL(V)$. Once one gets used to this, it is not as bad as it sounds: in practice, one often thinks of a representation as being the shared data of the vector space V and the linear action of G on that vector space.*

[7] This is in fact *equivalent* to the $G = \{1, X\}$ as they are related by a unitary transformation. More on equivalent transformations in a bit.

- The trivial representation $\{I\}$ can also of course be defined on a vector space of any dimension.

Examples representations for $O(3)$. Consider $O(3)$ the group of orthogonal matrices in dimension $d = 3$. We recall that this is the set of all 3×3 matrices M such that $MM^T = I$.

- The simplest representation, called the fundamental representation, is simply the set of all 3×3 orthogonal matrices.
- The morphism $R(g) = \det(M) = \pm 1$ is a representation of $O(3)$ on the vector space $\mathbb{R}$ (indeed $\det(AB) = \det(A)\det(B)$).

Fundamental representation of continuous groups. All continuous groups have the a 'fundamental' representation where the matrices in the group and the matrices in the representation coincide ("up to change of basis")[8].

Adjoint representation. Another important representation that is possible for any group is the adjoint representation. Thus far we have considered representations that map vectors to vectors, it is also possible to consider representations that map matrices to matrices. Let $V = M_2(\mathbb{C})$ denote the set of 2×2 complex matrices. The linear super-operator

$$A \mapsto U_g A U_g^\dagger \tag{9.16}$$

where $U_g = R(g)$ is a possible representation of G. For example, $U...U^\dagger$ for $U \in SU(2)$ is a representation of $SU(2)$.

So far we have spotted the representations corresponding to a symmetry group just by 'seeing them'. In fact, as I discussed earlier, the process often in physics goes the other way around. We know the symmetry at the level of the representation and then abstractify to identify the underlying group. But what about going the other way around - what if we have a group, and don't know any of its (non-trivial) representations, and want to find one?

Regular representation of finite groups. All finite groups admit what is known as the 'regular' representation as one of its representations.

Definition 9.3.2 (Regular representation)**.** For a finite group of order h, one can construct the so-called regular representation using $h \times h$ matrices as follows. First start from the following *reordered* Cayley table (here for $h = 3$):

$$C = \begin{array}{c|ccc} * & e & a^{-1} & b^{-1} \\ \hline e & e & a^{-1} & b^{-1} \\ a & a & e & ab^{-1} \\ b & b & ba^{-1} & e \end{array} \tag{9.17}$$

Now the representation can be done using the following matrices for $g \in G$: We use a matrix which is zero everywhere except for the position that corresponds to the group element in the Cayley table:

$$(R(g))_{ij} = \delta_{g,C_{ij}} \tag{9.18}$$

[8]Note that although the matrices between the group G and its representatives $\{R_g : g \in G\} \subseteq \mathrm{GL}(V)$ are identical, we think of the abstract group and its representatives as conceptually distinct.

With this definition, e is represented by the identity matrix $R(e) = I$. It is easy to check that these matrices indeed follow the group algebra. You'll work through some examples of this in the problem sheet.

It is also possible to construct representations from a simpler (set of) already known representations.

Equivalent representations. Consider a group G and a representation $R(g) \forall g \in G$. We define now $R'(g) = SR(g)S^{-1}$ where S can be any invertible matrix (in practise, in most cases we come accross, it will be a unitary matrix). This is a *similarity* transformation[9]. It is easy to see that similarity transformations of representations are still representations. It is straightforward to verify that $R'(g)$ is a representation of G (i.e., if $R(gh) = R(g)R(h)$ then $R'(gh) = SR(g)R(h)S^{-1} = SR(g)S^{-1}SR(h)S^{-1} = R'(g)R'(h)$).

Definition 9.3.3 (Equivalent representation)**.** Two representations D and D' are equivalent if they are related by a similarity transformation $R'(g) = SR(g)S^{-1}$.

Roughly speaking, representations are equivalent if we can transform one to the other by a linear invertible transformation. If what follows, we shall be mainly concerned by unitary representations and transformations. In this case $SS^\dagger = 1$ and $S^\dagger = S^{-1}$. This means that we shall consider two representations as equivalent if they simply correspond to a change of basis: $R'(g) = UR(g)U^\dagger$.

Tensor product representation. For example, consider two representations R_1 and R_2 for a group G, it is straightforward to verify (*check this!*) that the tensor product of their representations $R_1 \otimes R_2$, i.e. the set of matrices such that

$$R_1(g) \otimes R_2(g) \tag{9.19}$$

for each element g, is also a representation. For example, $\{I \otimes I, Z \otimes Z\}$ is a representation of Z_2 (in fact, $\{I^{\otimes k}, Z^{\otimes k}\}$ is a representation for any k).

Tensor product representations are fundamental in physics whenever we take the symmetry property of a single system and want to study the properties of a composite system. For example, suppose we have a system of n particles each of which are $SU(2)$ symmetric. In this case, we will be interested in the representation of $SU(2)$ on $(C^2)^{\otimes n}$, and so a natural choice is $SU(2)^{\otimes n}$.

Direct sum representation. Another useful composite representation, one that plays a key role in physics, is the direct sum representation.

Definition 9.3.4. Consider two representations R_1, R_2 of a group G acting on vector space V_1, V_2. The direct sum $R_1 \oplus R_2$ is a representation of G acting on $V_1 \oplus V_2$ defined by

$$(R_1 \oplus R_2)(g)(v_1, v_2) := (R_1(g)v_1, R_2(g)v_2), \quad \text{for all } g \in G. \tag{9.20}$$

Or, writing the matrices out explicitly, $R_1 \oplus R_2$ acting on $V_1 \oplus V_2$ we have:

$$(R_1 \oplus R_2)(g) := \begin{pmatrix} R_1(g) & 0 \\ 0 & R_2(g) \end{pmatrix}, \quad \text{for all } g \in G. \tag{9.21}$$

[9]In linear algebra, two $n \times n$ matrices A and B are called similar if there exists an invertible n-by-n matrix P such that $B = P^{-1}AP$.

That this is indeed a representation follows straightforwardly from the block structure of Eq. (9.21). (If this isn't immediately clear to you, do work through it explicitly). We can also take the direct sum of the same representation, i.e., $R_1 \oplus R_1$, in which case we say that R_1 has multiplicity of two, and we write

$$(R_1 \oplus R_1)(g) = \begin{pmatrix} R_1(g) & 0 \\ 0 & R_1(g) \end{pmatrix} = I \otimes R_1(g), \quad \text{for all } g \in G. \tag{9.22}$$

Notice that due to the block structure of a direct sum representation the action of an element of the representation structure of a group leave certain subspaces invariant. This will turn out to be very important.

Hopefully it is now clear how you can take simple representations of a group and create more complex ones. In many cases, we will in fact be more interested in going in the other direction. Taking a complex representation and trying to break it down into a simpler one. More concretely, one of the things representation theory is most useful for is *taking a representation (e.g. say a tensor one), and expressing it as a direct sum of representations on smaller subspaces.* We will discuss this in Section 9.4